

# Two CSCS-based iteration methods for absolute value equations involving Toeplitz matrix

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## Abstract

Recently, two kinds of HSS-based iteration methods are constructed for coping with the absolute value equation (AVE), which is a family of non-differentiable NP-hard problem. In present paper, we focus on developing the CSCS-based methods for solving the absolute value equation (AVE) involving the Toeplitz matrix, and propose the Picard-CSCS method and the nonlinear CSCS-like iterative method. With the help of introducing a smoothing approximate function, we give some theoretical analyses for the convergence of the CSCS-based iteration methods for the AVE. The advantage of these methods is that they do not require storage of coefficient matrix, and the linear sub-systems can be solved efficiently via the fast Fourier transform (FFT). Therefore, computational cost and computer storage may be saved in actual implementations. Extensive numerical experiments involving the numerical solutions of fractional diffusion equations are employed to demonstrate the robustness and effectiveness of the proposed methods and to compare with the recent methods.

**Key words:** Absolute value equation; CSCS-based iteration; Toeplitz matrix; Convergence analysis; Smoothing approximate function; Fast Fourier transform

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## 1 Introduction

In this study, we are interested in the AVE with non-Hermitian Toeplitz matrix of the form

$$Ax - |x| = b, \quad A \in \mathbb{C}^{n \times n}, \quad x, b \in \mathbb{C}^n \quad (1)$$

where  $|x|$  denotes the component-wise absolute value of the vector  $x$ . A slightly more generalized form of the AVE,

$$Ax - B|x| = b, \quad A \in \mathbb{C}^{m \times n}, \quad B \in \mathbb{C}^{m \times n}, \quad x, b \in \mathbb{C}^m \quad (2)$$

was discussed in [1] and investigated in a more general context in [2, 3]. Moreover, the theoretical and numerical aspects of these problems have been extensively investigated in recent literature [2, 4–7].

Generally speaking, the AVE (1) arises from quadratic programs, linear programs, bimatrix games and other problems, which can all be resulted in an linear complementarity problem (LCP) [7, 8], and

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the LCP is equivalent to the AVE (1). This means that the AVE is NP-hard in its general form [2, 5, 7]. If  $B = 0$ , then generalized AVE (2) reduces to a system of linear equations  $A\mathbf{x} = \mathbf{b}$ , which have several applications in scientific computation [7].

The recent research concerning the AVE contents can be summarized as the following aspects, one is the theoretical analysis, which focuses on the theorem of alternatives, various equivalent reformulations, and the existence and nonexistence of solutions; refer to [1, 3, 4, 8] for details. And the other is how to solve the AVE numerically. In the last decade, based on the fact that the LCP can be reduced to the AVE, which enjoys a very special and simple structure, a large variety of numerical methods for solving the AVE (1) can be found in the recent literature; see e.g. [6, 7, 9–11] and references therein. For example, a finite computational algorithm that is solved by a finite succession of linear programs (SLP) in [5], and a semi-smooth Newton method is proposed in [12], which largely shortens the computation time than the SLP method. Furthermore, a smoothing Newton algorithm was presented in [8], which was proved to be globally convergent and the convergence rate was quadratic under the condition that the singular values of  $A$  exceed 1. This condition was weaker than the one applied in [12].

During recent years, the Picard-HSS iteration method and nonlinear HSS-like method are established to solve the AVE in succession [13, 14], respectively. The sufficient conditions to guarantee the convergence of this method and some numerical experiments are given to show the effectiveness of the method. However, the numbers of the inner HSS iterative steps are often problem-dependent and difficult to be determined in actual computations. Moreover, the iterative vector can not be updated timely. It has shown that the nonlinear HSS-like iterative method is more efficient than the Picard-HSS iteration method in aspects of the defect mentioned above, which is designed originally for solving weakly nonlinear systems in [15]. In order to accelerate the nonlinear HSS-like iteration method, Zhang [16] had extended the preconditioned HSS (PHSS) method [17] to solve the AVE and also exploit the relaxation technique to accelerate his proposed methods. Meanwhile, numerical results also show the effectiveness of his proposed method in [16]. In this paper, we consider the special case of  $A$  involving the non-Hermitian Toeplitz structure. Similar to the strategies of [13, 14], two kinds of circulant and skew-circulant splitting (CSCS)-based methods are proposed to fast solve the AVE (1).

The rest of this paper is organized as follows. In Section 2 we review the CSCS iteration method and its relative topics. In section 3, we devote to introduce two CSCS-based iteration methods to solve AVE (1) and investigate their convergence properties, respectively. Numerical experiments are reported in Section 4, to shown the feasibility and effectiveness of the CSCS-based methods. Finally, the paper closes with some conclusions in Section 5.

## 2 The CSCS method

Here let  $A \in \mathbb{C}^{n \times n}$  be a non-Hermitian Toeplitz matrix of the following form

$$A = \begin{bmatrix} a_0 & a_{-1} & \cdots & a_{2-n} & a_{1-n} \\ a_1 & a_0 & a_{-1} & \cdots & a_{2-n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n-2} & \cdots & a_1 & a_0 & a_{-1} \\ a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \end{bmatrix},$$

i.e.,  $A$  is constant along its diagonals; see [18], and  $B \in \mathbb{C}^{n \times n}$  be a zero matrix, The general AVE (2) reduced to the system of linear equations

$$A\mathbf{x} = \mathbf{b}. \quad (3)$$

It is well-known that a Toeplitz matrix  $A$  possesses a circulant and skew-circulant splitting [19]

$A = C + S$ , where

$$C = \frac{1}{2} \begin{bmatrix} a_0 & a_{-1} + a_{n-1} & \cdots & a_{2-n} + a_2 & a_{1-n} + a_1 \\ a_1 + a_{1-n} & a_0 & \cdots & \cdots & a_{2-n} + a_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n-2} + a_2 & \cdots & \cdots & a_0 & a_{-1} + a_{n-1} \\ a_{n-1} + a_{-1} & a_{n-2} + a_{-2} & \cdots & a_1 + a_{1-n} & a_0 \end{bmatrix}, \quad (4)$$

and

$$S = \frac{1}{2} \begin{bmatrix} a_0 & a_{-1} - a_{n-1} & \cdots & a_{2-n} - a_2 & a_{1-n} - a_1 \\ a_1 - a_{1-n} & a_0 & \cdots & \cdots & a_{2-n} - a_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n-2} - a_2 & \cdots & \cdots & a_0 & a_{-1} - a_{n-1} \\ a_{n-1} - a_{-1} & a_{n-2} - a_{-2} & \cdots & a_1 - a_{1-n} & a_0 \end{bmatrix}, \quad (5)$$

Note that  $C$  is a circulant matrix and  $S$  is a skew-circulant matrix. A circulant matrix can be diagonalized by the discrete Fourier matrix  $F$  and a skew-circulant matrix can be diagonalized by a discrete Fourier matrix with diagonal scaling, i.e.,  $\hat{F} = F\Omega$ . That is to say, it holds that

$$FCF^* = \Lambda_C, \quad \hat{F}S\hat{F}^* = \Lambda_S, \quad (6)$$

where

$$F = (F)_{j,k} = \frac{1}{\sqrt{n}} e^{\frac{2\pi\iota}{n}jk}, \quad 0 \leq j, k \leq n-1, \quad \Omega = \text{diag}\left(1, e^{-\frac{\pi\iota}{n}}, \dots, e^{-\frac{(n-1)\pi\iota}{n}}\right),$$

and  $\iota$  is the imaginary unit [18, pp. 37-39].  $\Lambda_C$  and  $\Lambda_S$  are diagonal matrices formed by the eigenvalues of  $C$  and  $S$ , respectively, which can be obtained in  $\mathcal{O}(n \log n)$  operations by using the FFT. Moreover, Ng [20] established the following CSCS iteration method to solve non-Hermitian Toeplitz system of linear equations (3).

**Algorithm 1 The CSCS iteration method.**

Given an initial guess  $\mathbf{x}^{(0)} \in \mathbb{C}^n$ , compute  $\mathbf{x}^{(k)}$  for  $k = 0, 1, 2, \dots$  using the following iterative scheme until  $\{\mathbf{x}^{(k)}\}_{k=0}^\infty$  converges,

$$\begin{cases} (\sigma I + C)\mathbf{x}^{(k+\frac{1}{2})} = (\sigma I - S)\mathbf{x}^{(k)} + \mathbf{b}, \\ (\sigma I + S)\mathbf{x}^{(k+1)} = (\sigma I - C)\mathbf{x}^{(k+\frac{1}{2})} + \mathbf{b}, \end{cases}$$

where  $\sigma$  is a positive constant and  $I$  is the identity matrix.

In the matrix-vector form, the CSCS iteration can be equivalently rewritten as

$$\mathbf{x}^{(k+1)} = \mathcal{M}(\sigma)\mathbf{x}^{(k)} + \mathcal{G}(\sigma)\mathbf{b} = (\mathcal{M}(\sigma))^{k+1}\mathbf{x}^{(0)} + \sum_{j=0}^k (\mathcal{M}(\sigma))^j \mathcal{G}(\sigma)\mathbf{b}, \quad k = 0, 1, 2, \dots,$$

where

$$\mathcal{M}(\sigma) = (\sigma I + S)^{-1}(\sigma I - C)(\sigma I + C)^{-1}(\sigma I - S) \text{ and } \mathcal{G}(\sigma) = 2\sigma(\sigma I + S)^{-1}(\sigma I + C)^{-1}.$$

It is easy to see that CSCS is a stationary iterative method obtained from the splitting

$$A = \mathcal{B}(\sigma) - \mathcal{C}(\sigma),$$

where

$$\mathcal{B}(\sigma) = \frac{1}{2\sigma}(\sigma I + C)(\sigma I + S) \quad \text{and} \quad \mathcal{C}(\sigma) = \frac{1}{2\sigma}(\sigma I - C)(\sigma I - S).$$

On the other hand, we have

$$\mathcal{M}(\sigma) = (\mathcal{B}(\sigma))^{-1}\mathcal{C}(\sigma) \quad \text{and} \quad \mathcal{G}(\sigma) = (\mathcal{B}(\sigma))^{-1}.$$

Here,  $\mathcal{M}(\sigma)$  is the iterative matrix of the CSCS method. We remark that the CSCS iteration method is greatly similar to the HSS iteration method [21] and its variants, see e.g. [22].

When the circulant part  $C$  and skew-circulant part  $S$  of the coefficient matrix  $A \in \mathbb{C}^{n \times n}$  are both positive definite, Ng proved that the spectral radius  $\rho(\mathcal{M}(\sigma))$  of the CSCS iterative matrix  $\mathcal{M}(\sigma)$  is less than 1 for any positive iterative parameters  $\sigma$ , i.e., the CSCS iteration method unconditionally converges to the exact solution of  $A\mathbf{x} = \mathbf{b}$  for any initial guess  $\mathbf{x}^{(0)} \in \mathbb{C}^n$ ; refer to [20] for details.

### 3 Two CSCS-based methods for AVEs

Motivated by the pioneer works of [13,14], we extend the classical CSCS iteration method to two types of CSCS-based methods for solving the AVE (1). These methods will fully exploit the Toeplitz structure to accelerate the computation speed and save storage. Next, we will devote to constructing these two new methods, i.e., the Picard-CSCS iterative method and nonlinear CSCS-like iterative method.

#### 3.1 The Picard-CSCS iterative method

Recalling that the Picard iterative method is a fixed-point iterative method and the linear term  $A\mathbf{x}$  and the nonlinear term  $|\mathbf{x}| + \mathbf{b}$  are separated [13,14], the AVE can be solved by using of the Picard iterative method

$$A\mathbf{x}^{(k+1)} = |\mathbf{x}^{(k)}| + \mathbf{b}, \quad k = 0, 1, 2, \dots \quad (7)$$

We assume that the Toeplitz matrix  $A$  is non-Hermitian positive definite. In this case, the next iterate of  $\mathbf{x}^{(k+1)}$  can be approximately computed by the CSCS iteration by making use of  $A = \mathcal{B}(\sigma) - \mathcal{C}(\sigma)$  as following (see [23])

$$\mathcal{B}(\sigma)\mathbf{x}^{(k,\ell+1)} = \mathcal{C}(\sigma)\mathbf{x}^{(k,\ell)} + |\mathbf{x}^{(k)}| + \mathbf{b}, \quad \ell = 0, 1, \dots, l_k - 1, \quad k = 0, 1, 2, \dots, \quad (8)$$

where  $\mathcal{B}(\sigma)$  and  $\mathcal{C}(\sigma)$  are the matrices defined in the previous section,  $\sigma$  is a positive constant,  $\{l_k\}_{k=0}^{\infty}$  a prescribed sequence of positive integers, and  $\mathbf{x}^{(k,0)} = \mathbf{x}^{(k)}$  is the starting point of the inner CSCS iteration at  $k$ th outer Picard iteration. This leads to the inexact Picard iteration method, called Picard-CSCS iteration method, for solving the system (1) which can be summarized as following (refer to [23]).

#### Algorithm 2 The Picard-CSCS iteration method

Let  $A = C + S \in \mathbb{C}^{n \times n}$  be a non-Hermitian Toeplitz matrix;  $C$  and  $S$  are the circulant and skew-circulant parts of  $A$  given in (4) and (5) and they are both positive definite. Given an initial guess  $\mathbf{x}^{(0)} \in \mathbb{C}^n$  and a sequence  $\{l_k\}_{k=0}^{\infty}$  of positive integers, compute  $\mathbf{x}^{(k+1)}$  for  $k = 0, 1, 2, \dots$ , using the following iteration scheme until  $\{\mathbf{x}^{(k)}\}$  satisfies the following stopping criterion:

- (a) Set  $\mathbf{x}^{(k,0)} = \mathbf{x}^{(k)}$ ;
- (b) For  $\ell = 0, 1, \dots, l_k - 1$ , solve the following linear systems to obtain  $\mathbf{x}^{(k,\ell+1)}$ :

$$\begin{cases} (\sigma I + C)\mathbf{x}^{(k,\ell+\frac{1}{2})} = (\sigma I - S)\mathbf{x}^{(k,\ell)} + |\mathbf{x}^{(k)}| + \mathbf{b}, \\ (\sigma I + S)\mathbf{x}^{(k,\ell+1)} = (\sigma I - C)\mathbf{x}^{(k,\ell+\frac{1}{2})} + |\mathbf{x}^{(k)}| + \mathbf{b}, \end{cases}$$

where  $\sigma$  is a given positive constant.

- (c) Set  $\mathbf{x}^{(k+1)} := \mathbf{x}^{(k,l_k)}$ .

The advantage of Picard-CSCS iterative method is obvious. First, the two linear subsystems in all inner CSCS iterations have the same shifted circulant coefficient matrix  $\sigma I + C$  and shifted skew-circulant coefficient matrix  $\sigma I + S$ , which are constant with respect to the iteration index  $k$ . Second, the exact solutions can be efficiently achieved via FFTs in  $\mathcal{O}(n \log n)$  operations [20, 24]. Hence, the computations of the Picard-CSCS iteration method could be much cheaper than that of the Picard-HSS iteration method.

The next theorem provides sufficient conditions for the convergence of the Picard-CSCS method to solve system (1).

**Theorem 1** *Let  $A = C + S \in \mathbb{C}^{n \times n}$  be a non-Hermitian Toeplitz matrix;  $C$  and  $S$  are the circulant and skew-circulant parts of  $A$  given in (4) and (5) and they are both positive definite. Let also  $\eta = \|A^{-1}\|_2 < 1$ . Then the AVE (2) has a unique solution  $\mathbf{x}^*$ , and for any initial guess  $\mathbf{x}^{(0)} \in \mathbb{C}^n$  and any sequence of positive integers  $\ell_k, k = 0, 1, 2, \dots$ , the iteration sequence  $\{\mathbf{x}^{(k)}\}_{k=0}^\infty$  produced by the Picard-CSCS iteration method converges to  $\mathbf{x}^*$  provided that  $l = \liminf_{k \rightarrow \infty} \ell_k \geq N$ , where  $N$  is a natural number satisfying*

$$\left\| (\mathcal{M}(\sigma))^s \right\|_2 < \frac{1 - \eta}{1 + \eta}, \quad \forall s \geq N.$$

**Proof.** The proof uses arguments similar to those in the proof of the convergence theorem of the Picard-HSS iteration method; see [13, 15]. In fact, we only need to replace the Hermitian matrix  $H$  and the skew-Hermitian matrix  $S$  of the convergence theorem of the Picard-CSCS iteration method by the circulant matrix  $C$  and the skew-circulant matrix  $S$ , and then obtain the convergence theorem of the Picard-CSCS iteration method.

According to Theorem 1, we see that the Picard-CSCS iteration method to solve the AVE (2) is convergent if the matrix  $(M - I)^{-1}(M + I)$  is positive definite,  $\eta = \|(M + I)^{-1}(M - I)\|_2 < 1$  (see [13] for the definition of  $M$ ) and the sequence  $\ell_k, k = 0, 1, 2, \dots$ , is defined as in Theorem 1. Similar to [13], the residual-updating form of the Picard-CSCS iteration method can be written as following.

**Algorithm 3 The Picard-CSCS iteration method** (residual-updating variant)

*Let  $A = C + S \in \mathbb{C}^{n \times n}$  be a non-Hermitian Toeplitz matrix;  $C$  and  $S$  are the circulant and skew-circulant parts of  $A$  given in (4) and (5) and they are both positive definite. Given an initial guess  $\mathbf{x}^{(0)} \in \mathbb{C}^n$  and a sequence  $\{\ell_k\}_{k=0}^\infty$  of positive integers, compute  $\mathbf{x}^{(k+1)}$  for  $k = 0, 1, 2, \dots$ , using the following iteration scheme until  $\{\mathbf{x}^{(k)}\}$  satisfies the following stopping criterion:*

- (a) Set  $\mathbf{s}^{(k,0)} = \mathbf{0}$  and  $\mathbf{r}^{(k)} = |\mathbf{x}^{(k)}| + \mathbf{b} - A\mathbf{x}^{(k)}$ ;
- (b) For  $\ell = 0, 1, \dots, \ell_k - 1$ , solve the following linear systems to obtain  $\mathbf{s}^{(k,\ell+1)}$ :

$$\begin{cases} (\sigma I + C)\mathbf{s}^{(k,\ell+\frac{1}{2})} = (\sigma I - S)\mathbf{s}^{(k,\ell)} + \mathbf{r}^{(k)}, \\ (\sigma I + S)\mathbf{s}^{(k,\ell+1)} = (\sigma I - C)\mathbf{s}^{(k,\ell+\frac{1}{2})} + \mathbf{r}^{(k)}, \end{cases}$$

where  $\sigma$  is a given positive constant.

- (c) Set  $\mathbf{x}^{(k+1)} := \mathbf{x}^{(k)} + \mathbf{s}^{(k,\ell_k)}$ .

### 3.2 The nonlinear CSCS-like iteration method

In the Picard-CSCS iteration, the numbers  $\ell_k, k = 0, 1, 2, \dots$  of the inner CSCS iterative steps are often problem-dependent and difficult to be determined in actual computations [13, 14, 23]. Moreover, the iterative vector can not be updated timely. Thus, to avoid the defection and still preserve the advantages of the Picard-CSCS iterative method, based on the nonlinear fixed-point equations

$$(\sigma I + C)\mathbf{x} = (\sigma I - S)\mathbf{x} + |\mathbf{x}| + \mathbf{b}, \quad \text{and} \quad (\sigma I + S)\mathbf{x} = (\sigma I - C)\mathbf{x} + |\mathbf{x}| + \mathbf{b},$$

we propose the following nonlinear CSCS-like iteration method.

**Algorithm 4 The nonlinear CSCS-like iteration method**

Let  $A = C + S \in \mathbb{C}^{n \times n}$  be a non-Hermitian Toeplitz matrix;  $C$  and  $S$  are the circulant and skew-circulant parts of  $A$  given in (4) and (5) and they are both positive definite. Given an initial guess  $\mathbf{x}^{(0)} \in \mathbb{C}^n$  and compute  $\mathbf{x}^{(k+1)}$  for  $k = 0, 1, 2, \dots$ , using the following iteration scheme until  $\{\mathbf{x}^{(k)}\}$  satisfies the following stopping criterion:

$$\begin{cases} (\sigma I + C)\mathbf{x}^{(k+\frac{1}{2})} = (\sigma I - S)\mathbf{x}^{(k)} + |\mathbf{x}^{(k)}| + \mathbf{b}, \\ (\sigma I + S)\mathbf{x}^{(k+1)} = (\sigma I - C)\mathbf{x}^{(k+\frac{1}{2})} + |\mathbf{x}^{(k+\frac{1}{2})}| + \mathbf{b}, \end{cases} \quad (9)$$

where  $\sigma$  is a given positive constant.

Define

$$\begin{cases} \mathcal{U}(\mathbf{x}) = (\sigma I + C)^{-1}((\sigma I - S)\mathbf{x} + |\mathbf{x}| + \mathbf{b}), \\ \mathcal{V}(\mathbf{x}) = (\sigma I + S)^{-1}((\sigma I - C)\mathbf{x} + |\mathbf{x}| + \mathbf{b}), \end{cases} \quad (10)$$

and

$$\Theta(\mathbf{x}) = \mathcal{V} \circ \mathcal{U}(\mathbf{x}) := \mathcal{V}(\mathcal{U}(\mathbf{x})).$$

Then the nonlinear CSCS-like iterative scheme can be equivalently expressed as

$$\mathbf{x}^{(k+1)} = \Theta(\mathbf{x}^{(k)}). \quad (11)$$

The Ostrowski theorem, i.e., Theorem 10.1.3 in [25], gives a local convergence theory about a one-step stationary nonlinear iteration. Based on this, Zhu and Zhang established the local convergence theory for the nonlinear CSCS-like iteration method in [23]. However, these convergence theory has a strict requirement that  $f(\mathbf{x}) = |\mathbf{x}| + \mathbf{b}$  is  $\mathcal{F}$ -differentiable at a point  $\mathbf{x}^* \in \mathbb{D}$  such that  $A\mathbf{x}^* - |\mathbf{x}^*| = \mathbf{b}$ . Obviously, the absolute value function  $|\mathbf{x}|$  is non-differentiable.

Leveraging the smoothing approximate function introduced in [26], we can establish the following local convergence theory for nonlinear CSCS-like iterative method. But firstly, we must review this smoothing approximation and its properties, which will be used in the next section.

Define  $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}^n$  by

$$\varphi(\mathbf{x}) = \frac{1}{\mu} \ln \left( \exp \left( \frac{\mathbf{x}}{\mu} \right) + \exp \left( \frac{-\mathbf{x}}{\mu} \right) \right), \quad \mathbf{x} \in \mathbb{C}^n. \quad (12)$$

It is clear that  $\varphi(\mathbf{x})$  is a smoothing function of  $|\mathbf{x}|$ , now we give some properties of  $\varphi(\mathbf{x})$ , which will be used in the next section.

**Lemma 1** ([26])  $\varphi(\mathbf{x})$  is a uniformly smoothing approximation function of  $|\mathbf{x}|$ , i.e.,

$$\|\varphi(\mathbf{x}) - |\mathbf{x}|\| \leq \sqrt{n} \ln 2 \cdot \mu.$$

**Lemma 2** ([26]) For any  $\mu > 0$ , the Jacobian of  $\varphi(\mathbf{x})$  at  $\mathbf{x} = (x_j) \in \mathbb{C}^n$  is

$$D = \varphi'(\mathbf{x}) = \text{diag} \left( \frac{\exp \left( \frac{x_j}{\mu} \right) - \exp \left( \frac{-x_j}{\mu} \right)}{\exp \left( \frac{x_j}{\mu} \right) + \exp \left( \frac{-x_j}{\mu} \right)} \right), \quad x_j \in \mathbb{C}, \quad j = 1, 2, \dots, n. \quad (13)$$

**Lemma 3** Assume that  $\varphi : \mathbb{D} \subset \mathbb{C}^n \rightarrow \mathbb{C}^n$  is  $\mathcal{F}$ -differentiable at a point  $\mathbf{x}^* \in \mathbb{D}$  such that  $A\mathbf{x}^* = \varphi(\mathbf{x}^*) + \mathbf{b}$ . Suppose that  $C$  and  $S$  are the circulant and the skew-circulant parts of the matrix  $A = C + S$  given in (4) and (5), and  $C$  and  $S$  both are positive definite matrices. Denote by

$$\mathcal{M}(\alpha, \mathbf{x}^*) = (\sigma I + S)^{-1}(\sigma I - C + \varphi'(\mathbf{x}))(\sigma I + C)(\alpha I - S + \varphi'(\mathbf{x}))$$

and

$$\begin{aligned} \delta &= \max\{\|\varphi'(\mathbf{x}^*)(\sigma I + S)^{-1}\|, \|\varphi'(\mathbf{x}^*)(\sigma I + C)^{-1}\|\}, \\ \xi &= \max\{\|(\sigma I - S)(\sigma I + S)^{-1}\|, \|(\sigma I - C)(\sigma I + C)^{-1}\|\}, \end{aligned}$$

then  $\rho(\mathcal{M}(\sigma; \mathbf{x}^*)) < 1$  holds; in other word,  $\mathbf{x} \in \mathbb{D} \subset \mathbb{C}^n$  is a point of attraction of the nonlinear CSCS-like iteration, provided  $\delta < 1 - \xi$ .

Leveraging the smoothing approximate function  $\varphi(\mathbf{x})$  in (11), we define

$$\begin{cases} \bar{\mathcal{U}}(\mathbf{x}) = (\sigma I + C)^{-1}((\sigma I - S)\mathbf{x} + \varphi(\mathbf{x}) + \mathbf{b}), \\ \bar{\mathcal{V}}(\mathbf{x}) = (\sigma I + S)^{-1}((\sigma I - C)\mathbf{x} + \varphi(\mathbf{x}) + \mathbf{b}), \end{cases}$$

and

$$\bar{\Theta}(\mathbf{x}) = \bar{\mathcal{V}} \circ \bar{\mathcal{U}}(\mathbf{x}) := \bar{\mathcal{V}}(\bar{\mathcal{U}}(\mathbf{x})).$$

Then the nonlinear CSCS-like iterative scheme can be equivalently expressed as

$$\bar{\mathbf{x}}^{(k+1)} = \bar{\Theta}(\mathbf{x}^{(k)}). \quad (14)$$

**Theorem 2** *Assume that the condition of Lemma 3 are satisfied,  $C$  and  $S$  be circulant and skew-circulant parts of the Toeplitz matrix  $A$ , respectively. For any initial guess  $\mathbf{x}^{(0)} \in \mathbb{C}^n$ , the iteration sequence  $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$  produced by the nonlinear CSCS-like iteration method can be instead approximately by that produced by its smoothed nonlinear CSCS-like iterative scheme (14), i.e.,*

$$\|\Theta(\mathbf{x}^{(k)}) - \bar{\Theta}(\mathbf{x}^{(k)})\| \leq \varepsilon, \quad \text{for } \forall \varepsilon > 0,$$

provided

$$\mu \leq \frac{\|(\sigma I + S)\| \varepsilon}{(2 + \|(\sigma I + C)^{-1}\|) \cdot \sqrt{n} \ln 2}.$$

**Proof.** First, we give the well-known inequality  $||s| - |t|| \leq |s - t|$ ,  $s, t \in \mathbb{C}$  and the result  $\|(\sigma I - C)(\sigma I + C)^{-1}\| < 1$  achieved in [20]. Then based on iterative scheme (11) and (14), we obtain

$$\begin{aligned} \|\bar{\mathbf{x}}^{(k+1)} - \mathbf{x}^{(k+1)}\| &= \|\bar{\Theta}(\mathbf{x}^{(k)}) - \Theta(\mathbf{x}^{(k)})\| \\ &\leq \|(\sigma I + S)^{-1}(\sigma I - C)(\sigma I + C)^{-1}(\varphi(\mathbf{x}) - |\mathbf{x}|)\| + \|(\sigma I + S)^{-1}(\varphi(\bar{\mathcal{U}}(\mathbf{x})) - |\mathcal{U}(\mathbf{x})|)\| \\ &\leq \|(\sigma I + S)^{-1}(\sigma I - C)(\sigma I + C)^{-1}\| \cdot \|\varphi(\mathbf{x}) - |\mathbf{x}|\| \\ &\quad + \|(\sigma I + S)^{-1}(\varphi(\bar{\mathcal{U}}(\mathbf{x})) - |\bar{\mathcal{U}}(\mathbf{x})| + |\bar{\mathcal{U}}(\mathbf{x})| - |\mathcal{U}(\mathbf{x})|)\| \\ &\leq \|(\sigma I + S)^{-1}(\sigma I - C)(\sigma I + C)^{-1}\| \cdot \|\varphi(\mathbf{x}) - |\mathbf{x}|\| \\ &\quad + \|(\sigma I + S)^{-1}\| \cdot \|\varphi(\bar{\mathcal{U}}(\mathbf{x})) - |\bar{\mathcal{U}}(\mathbf{x})|\| + \|(\sigma I + S)^{-1}\| \cdot \|\bar{\mathcal{U}}(\mathbf{x}) - \mathcal{U}(\mathbf{x})\| \\ &= \|(\sigma I + S)^{-1}\| \cdot (2\sqrt{n} \ln 2 \cdot \mu + \|(\sigma I + C)^{-1}\| \cdot \|\varphi(\mathbf{x}) - |\mathbf{x}|\|) \\ &= \frac{(2 + \|(\sigma I + C)^{-1}\|) \cdot \sqrt{n} \ln 2 \cdot \mu}{\|(\sigma I + S)\|} \end{aligned}$$

For  $\forall \varepsilon > 0$ ,  $\|\bar{\mathbf{x}}^{(k+1)} - \mathbf{x}^{(k+1)}\| = \|\bar{\Theta}(\mathbf{x}^{(k)}) - \Theta(\mathbf{x}^{(k)})\| \leq \varepsilon$  holds, provided

$$\mu \leq \frac{\|(\sigma I + S)\| \varepsilon}{(2 + \|(\sigma I + C)^{-1}\|) \cdot \sqrt{n} \ln 2}.$$

This completes the proof.  $\square$

**Theorem 3** *Assume that the conditions of Theorem 2 are satisfied. Denoted by*

$$\delta = \max\{\|(\sigma I + C)^{-1}\|_2, \|(\sigma I + S)^{-1}\|_2\},$$

and

$$\xi = \max\{\|(\sigma I - S)(\sigma I + S)^{-1}\|_2, \|(\sigma I - C)(\sigma I + C)^{-1}\|_2\},$$

Then the spectral radius  $\rho(\mathcal{M}(\sigma, \mathbf{x}^*))$  of the matrix  $\mathcal{M}(\sigma, \mathbf{x}^*)$  is less than 1, where

$$\mathcal{M}(\sigma, \mathbf{x}^*) = (\sigma I + S)^{-1}(\sigma I - C + D)(\sigma I + C)^{-1}(\sigma I - S + D),$$

and  $D$  is the Jacobian of  $\varphi(\mathbf{x})$  at  $\mathbf{x}^* \in \mathbb{N}(\mathbf{x}^*) \subset \mathbb{D} \subset \mathbb{C}^n$  defined in (13), provided that

$$\delta < 1 - \xi. \quad (15)$$

That is to say, for any initial guess  $\mathbf{x}^{(0)} \in \mathbb{D} \subset \mathbb{C}^n$ , the iteration sequence  $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$  produced by the nonlinear CSCS-like iteration method converges to  $\mathbf{x}^*$ , or  $\mathbf{x}^*$  is a point of attraction of the nonlinear CSCS-like iteration, provided the condition (15).

**Proof.** For  $\forall \varepsilon > 0$ , We only need to prove

$$\|\mathbf{x}^{(k+1)} - \mathbf{x}^*\| \leq \|\mathbf{x}^{(k+1)} - \bar{\mathbf{x}}^{(k+1)}\| + \|\bar{\mathbf{x}}^{(k+1)} - \mathbf{x}^*\| \leq \|\Theta(\mathbf{x}^{(k)}) - \bar{\Theta}(\mathbf{x}^{(k)})\| + \|\bar{\Theta}(\mathbf{x}^{(k)}) - \mathbf{x}^*\| \leq \varepsilon, \quad (16)$$

where  $\Theta(\mathbf{x}^{(k)})$  is defined in (11) and  $\bar{\Theta}(\mathbf{x}^{(k)})$  is defined in (14).

Via using the Theorem 2, the former part  $\|\Theta(\mathbf{x}^{(k)}) - \bar{\Theta}(\mathbf{x}^{(k)})\| \leq \varepsilon$  holds for  $\forall \varepsilon > 0$ , provided

$$\mu \leq \frac{\|(\sigma I + S)\| \varepsilon}{(2 + \|(\sigma I + C)^{-1}\|) \cdot \sqrt{n} \ln 2}.$$

As the uniformly smoothing approximation function  $\varphi(\mathbf{x})$  of  $|\mathbf{x}|$  is  $\mathcal{F}$ -differentiable at a point  $\mathbf{x}^* \in \mathbb{D}$  such that  $A\mathbf{x}^* - |\mathbf{x}^*| = \mathbf{b}$ , according Lemma 3,  $\mathbf{x}^*$  is a point of attraction of the nonlinear CSCS-like iteration, that is the second part in (16)

$$\|\bar{\mathbf{x}}^{(k+1)} - \mathbf{x}^*\| = \|\bar{\Theta}(\mathbf{x}^{(k)}) - \mathbf{x}^*\| \leq \varepsilon$$

holds for  $\forall \varepsilon$ , provided  $\rho(\mathcal{M}(\sigma; \mathbf{x}^*)) < 1$ .

Next we prove  $\rho(\mathcal{M}(\sigma; \mathbf{x}^*)) < 1$ . Via straightforward computations we have

$$\begin{aligned} (\sigma I + S)\mathcal{M}(\sigma; \mathbf{x}^*)(\sigma I + S)^{-1} &= (\sigma I + S)\mathcal{M}(\sigma)(\sigma I + S)^{-1} + (\sigma I - C)(\sigma I + C)^{-1}D(\sigma I + S)^{-1} \\ &\quad + D(\sigma I + C)^{-1}(\sigma I - S)(\sigma I + S)^{-1} + D(\sigma I + C)^{-1}D(\sigma I + S)^{-1}, \end{aligned}$$

where  $D$  is the Jacobian of the smoothing approximation function  $\varphi(\mathbf{x})$  at  $\mathbf{x}^*$ , also since

$$\|\mathcal{M}(\sigma)\|_2 \leq \|(\sigma I - C)(\sigma I + C)^{-1}\|_2 \cdot \|(\sigma I - S)(\sigma I + S)^{-1}\|_2 \leq \xi^2.$$

We obtain

$$\begin{aligned} \|\mathcal{M}(\sigma; \mathbf{x}^*)\|_2 &= \|(\sigma I + S)\mathcal{M}(\sigma; \mathbf{x}^*)(\sigma I + S)^{-1}\|_2 \\ &\leq \|(\sigma I + S)\mathcal{M}(\sigma)(\sigma I + S)^{-1}\|_2 + \|(\sigma I - C)(\sigma I + C)^{-1}D(\sigma I + S)^{-1}\|_2 \\ &\quad + \|D(\sigma I + C)^{-1}(\sigma I - S)(\sigma I + S)^{-1}\|_2 + \|D(\sigma I + C)^{-1}D(\sigma I + S)^{-1}\|_2 \\ &\leq \|\mathcal{M}(\sigma)\|_2 + \|(\sigma I - C)(\sigma I + C)^{-1}\|_2 \cdot \|D(\sigma I + S)^{-1}\|_2 \\ &\quad + \|D(\sigma I + C)^{-1}\|_2 \cdot \|(\sigma I - S)(\sigma I + S)^{-1}\|_2 + \|D(\sigma I + C)^{-1}\|_2 \cdot \|D(\sigma I + S)^{-1}\|_2 \\ &\leq \|\mathcal{M}(\sigma)\|_2 + \|(\sigma I - C)(\sigma I + C)^{-1}\|_2 \cdot \|(\sigma I + S)^{-1}\|_2 \\ &\quad + \|(\sigma I + C)^{-1}\|_2 \cdot \|(\sigma I - S)(\sigma I + S)^{-1}\|_2 + \|(\sigma I + C)^{-1}\|_2 \cdot \|(\sigma I + S)^{-1}\|_2 \\ &\leq \xi^2 + 2\xi\delta + \delta^2 = (\xi + \delta)^2. \end{aligned}$$

Now, under the condition  $\delta < 1 - \xi$ , we easily obtain  $\rho(\mathcal{M}(\sigma; \mathbf{x}^*)) \leq \|\mathcal{M}(\sigma; \mathbf{x}^*)\| < 1$ .  $\square$



**Remark 1.** An attractive feature of the nonlinear CSCS-like iterative method is that it avoids the use of the differentiable in actual iterative scheme, although we employ it in the convergence analysis. Thus, the smoothing approximate function  $\varphi(\mathbf{x})$  in (12) is not necessary in actual implementation.

At the end of this subsection, we remark that the main steps in nonlinear CSCS-like iteration method can be alternatively reformulated into residual-updating form similar to those in the Picard-CSCS iterative method as follows.

**Algorithm 5 (The nonlinear CSCS-like iteration method (residual-updating variant))**

Let  $A = C + S \in \mathbb{C}^{n \times n}$  be a non-Hermitian Toeplitz matrix;  $C$  and  $S$  are the circulant and skew-circulant parts of  $A$  given in (4) and (5) and they are both positive definite. Given an initial guess  $\mathbf{x}^{(0)} \in \mathbb{C}^n$  and compute  $\mathbf{x}^{(k+1)}$  for  $k = 0, 1, 2, \dots$ , using the following iteration scheme until  $\{\mathbf{x}^{(k)}\}$  satisfies the following stopping criterion:

$$\begin{cases} \mathbf{r}^{(k,0)} := |\mathbf{x}^{(k)}| + \mathbf{b} - A\mathbf{x}^{(k)}, \\ (\sigma I + C)\mathbf{r}^{(k,1)} = \mathbf{r}^{(k,0)}, & \mathbf{x}^{(k+\frac{1}{2})} = \mathbf{x}^{(k)} + \mathbf{r}^{(k,1)}, \\ \mathbf{r}^{(k,2)} := |\mathbf{x}^{(k+\frac{1}{2})}| + \mathbf{b} - A\mathbf{x}^{(k+\frac{1}{2})}, \\ (\sigma I + S)\mathbf{r}^{(k,3)} = \mathbf{r}^{(k,2)}, & \mathbf{x}^{(k+1)} = \mathbf{x}^{(k+\frac{1}{2})} + \mathbf{r}^{(k,3)}, \end{cases}$$

where  $\sigma$  is a given positive constant.

## 4 Numerical results

In this section, the numerical properties of the Picard-CSCS and the nonlinear CSCS-like methods are examined and compared experimentally by a suit of test problems. All the tests are performed in MATLAB R2014a on Intel(R) Core(TM) i5-3470 CPU @ 3.2 GHz and 8.00 GB of RAM, with machine precision  $10^{-16}$ , and terminated when the current residual satisfies

$$\frac{\|A\mathbf{x}^{(k)} - |\mathbf{x}^{(k)}| - \mathbf{b}\|_2}{\|\mathbf{b}\|_2} < 10^{-7},$$

where  $\mathbf{x}^{(k)}$  is the computed solution by each of the methods at iteration step  $k$ , and a maximum number of the iterations 200 is used.

In addition, the stopping criterion for the inner iterations of the Picard-CSCS method are set to be

$$\frac{\|\mathbf{b}^{(k)} - A\mathbf{s}^{(k,l_k)}\|_2}{\|\mathbf{b}^{(k)}\|_2} \leq \eta_k$$

where  $l_k$  is the number of the inner iteration steps and  $\eta_k$  is the prescribed tolerance for controlling the accuracy of the inner iterations at the  $k$ -th outer iteration step. If  $\eta_k$  is fixed for all  $k$ , then it is simply denoted by  $\eta$ .

In our numerical experiments, we use the zero vector as the initial guess, the accuracy of the inner iterations  $\eta_k$  for both Picard-CSCS and Picard-HSS iterative methods is fixed and set to 0.01, a maximum number of iterations 15 ( $l_k = 15, k = 0, 1, 2, \dots$ ) for inner iterations, and the right-hand side vector  $\mathbf{b}$  of AVEs (1) is taken in such a way that the vector  $\mathbf{x}_* = (x_1, x_2, \dots, x_n)^T$  with

$$x_k = (-1)^k \iota, \quad k = 1, 2, \dots, n$$

be the exact solution. The two sub-systems of linear equations involved are solved in the way if  $A\mathbf{x}_* = \mathbf{b}$ , then  $\mathbf{x}_* = A^{-1}\mathbf{b}$ . Moreover, if the two sub-systems of linear equations involved in the Picard-CSCS and the nonlinear CSCS-like iteration methods are solved by making use of the method presented in [24] and using parallel computing, the numerical results of the Picard-CSCS and the nonlinear CSCS-like iteration methods must be better.

In practical implementations, the optimal parameter  $\sigma_{\text{HSS}} = \sqrt{\lambda_{\max}\lambda_{\min}}$  recommended in [21] is employed for the Picard-HSS and the nonlinear HSS-like methods, where  $\lambda_{\min}$  and  $\lambda_{\max}$  are the minimum and the maximum eigenvalues of the Hermitian part  $H$  of the coefficient matrix  $A$ . Similarly, we adopt the optimal parameters  $\sigma_{\text{CSCS}}$  given in [20, Theorem 2] for the Picard-CSCS and the nonlinear CSCS-like methods. At the same time, it is remarkable that they only minimize the bound of the convergence factor of the iteration matrix, but not the spectral radius of the iteration matrix. Admittedly, the optimal parameters are crucial for guaranteeing fast convergence speeds of these parameter-dependent iteration methods, but they are generally very difficult to be determined, refer to e.g. [13, 15, 21, 23, 27] for a discussion of these issues.

To show that the proposed iteration methods can also be efficiently applied to deal with complex system of AVEs (1), we construct and test the following example, which is a Toeplitz system of AVEs with complex matrix.

**Example 1.** We consider that  $A \in \mathbb{C}^{n \times n}$  is a complex non-Hermitian, sparse and positive definite Toeplitz matrix with the following form

$$A = \begin{pmatrix} \gamma & c\iota & d\iota & & & & \\ -1-c\iota & \gamma & c\iota & d\iota & & & \\ -1-d\iota & -1-c\iota & \gamma & c\iota & d\iota & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & -1-d\iota & -1-c\iota & \gamma & c\iota & d\iota \\ & & & -1-d\iota & -1-c\iota & \gamma & c\iota \\ & & & & -1-d\iota & -1-c\iota & \gamma \end{pmatrix}, \quad (17)$$

where  $\iota = \sqrt{-1}$ . It means that the matrices  $A$  in the target AVEs are defined as Eq. (17). According to the performances of HSS-based methods, see [13, 14, 16], compared with other early established methods, we compare the proposed CSCS-based methods with HSS-based methods in Example 1. Then we will choose different parameters  $c$  and  $d$  and present the corresponding numerical results in Tables 2-3.

Table 1: The optimal parameters  $\sigma_{\text{opt}}^*$  for Example 1.

$\gamma$	$(c, d)$	$\sigma_{\text{opt}}^*$	$n = 128$	$n = 256$	$n = 512$	$n = 1024$	$n = 2048$	$n = 4096$
10	(2, 3)	$\sigma_{\text{HSS}}$	2.9710	2.9524	2.9477	2.9465	2.9462	2.9461
		$\sigma_{\text{CSCS}}$	1.1817	1.1818	1.1813	1.1813	1.1813	1.1813
13.5	(3, 4)	$\sigma_{\text{HSS}}$	3.6871	3.6595	3.6525	3.6507	3.6503	3.6502
		$\sigma_{\text{CSCS}}$	1.6008	1.5997	1.5989	1.5989	1.5988	1.5989

Table 2: Numerical results for Example 1 with different values of  $n$  ( $\gamma = 10$  and  $(c, d) = (2, 3)$ ).

Method		$n = 128$	$n = 256$	$n = 512$	$n = 1024$	$n = 2048$	$n = 4096$
Picard-HSS	IT_out	6	6	6	6	5	5
	IT_inn	9.8333	9.8333	9.5000	9.5000	9.2000	9.2000
	IT	59	59	57	57	46	46
	CPU	0.0158	0.0215	0.0243	0.0338	0.0511	0.1044
Picard-CSCS	IT_out	6	6	6	6	6	5
	IT_inn	6.3333	6.3333	6.0000	6.0000	6.0000	5.6000
	IT	38	38	36	36	36	28
	CPU	0.0112	0.0146	0.0189	0.0258	0.0431	0.0546
HSS-like	IT	37	36	35	34	33	31
	CPU	0.0136	0.0173	0.0218	0.0275	0.0531	0.1013
CSCS-like	IT	24	23	22	22	21	21
	CPU	0.0072	0.0097	0.0131	0.0215	0.0276	0.0514

Table 3: Numerical results for Example 1 with different values of  $n$  ( $\gamma = 13.5$  and  $(c, d) = (3, 4)$ ).

Method		$n = 128$	$n = 256$	$n = 512$	$n = 1024$	$n = 2048$	$n = 4096$
Picard-HSS	IT_out	5	5	5	5	5	5
	IT_inn	10.0000	10.0000	10.0000	10.0000	10.0000	10.0000
	IT	50	50	50	50	50	50
	CPU	0.0129	0.0161	0.0206	0.0298	0.0544	0.1135
Picard-CSCS	IT_out	5	5	5	5	5	5
	IT_inn	6.2000	6.0000	6.0000	6.0000	5.8000	5.8000
	IT	31	30	30	30	29	29
	CPU	0.0098	0.0123	0.0166	0.0248	0.0332	0.0581
HSS-like	IT	41	40	39	38	37	36
	CPU	0.0121	0.0174	0.0192	0.0306	0.0587	0.1131
CSCS-like	IT	24	23	23	22	21	21
	CPU	0.0075	0.0093	0.0136	0.0212	0.0289	0.0496

Firstly, the optimal parameters  $\sigma_{\text{CSCS}}$  and  $\alpha_{\text{HSS}}$  for Example 1 are listed in Table 1. It is remarkable that with the increase of the matrix dimension  $n$ , the optimal parameters  $\sigma_{\text{CSCS}}$  and  $\alpha_{\text{HSS}}$  are almost fixed or decreasing slightly. Moreover, in Tables 2-3, we report the numerical results with respect to the Picard-HSS, the nonlinear HSS-like, the Picard-CSCS and the nonlinear CSCS-like iterations. We also present the elapsed CPU time in seconds for the convergence (denoted as CPU) and the number of outer, inner and total iteration steps (outer and inner iterations only for both Picard-HSS and Picard-CSCS) for the convergence (denoted as IT\_out, IT\_inn and IT, respectively).

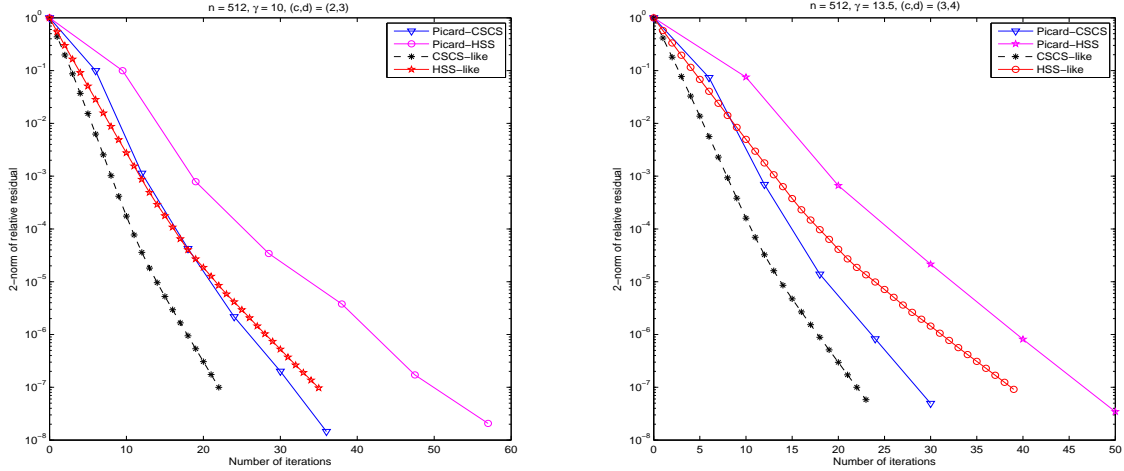


Figure 1: Convergence histories of the different iterative methods for two systems of AVEs with the size  $n = 512$  in Example 1.

According to Tables 2-3, it finds that these four iterative solvers, i.e., the Picard-HSS, the nonlinear HSS-like, the Picard-CSCS and the nonlinear CSCS-like iteration methods, can successfully archive approximate solutions to the system of AVEs for all different matrix dimensions. When the matrix dimension  $n$  is increasing, the number of outer and inner iteration steps are almost fixed for all iteration methods, and the number of total iteration steps show the similar phenomena. But the total CPU times for all iteration methods are increasing quickly. Moreover, in terms of outer iteration steps, the Picard-HSS and the Picard-CSCS have almost the same results, but the Picard-CSCS iteration method is better than the Picard-HSS iteration method in terms of inner iteration steps. Then as a results, the

Picard-CSCS iteration method is also more competitive than the Picard-HSS iteration method in aspects of the CPU elapsed time.

On the other hand, from Tables 2-3, we also observe that both the nonlinear CSCS-like and the Picard-CSCS iteration methods are better than the nonlinear HSS-like and the Picard-HSS iteration methods in terms of the number of iteration steps and the CPU elapsed time for solving the system of AVEs. In particular, the nonlinear CSCS-like method often enjoys the better performance than the Picard-CSCS method in our implementations. Moreover, the convergence histories of residual norms of these four different iterative algorithms are displayed in Fig. 1. In conclusion, the nonlinear CSCS-like iterative method is the best choice for coping with the AVEs concerning in Example 1. Besides, the Picard-CSCS iterative method can be regarded as an alternative.

**Example 2.** In order to evaluate the performances of the propose methods comprehensively, we consider a family of the practical AVE problems arising in numerical solutions of the following one-dimensional space nonlinear fractional reaction-diffusion equation, which is specially modified from Refs. [28, 29]:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_+ \frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha} + d_- \frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha} + |u(x,t)| + q(x,t), & x \in (0,1), \quad t \in [0,1], \\ u(0,t) = u(1,t) = 0, & 0 \leq t \leq 1, \\ u(x,0) = \phi(x), & 0 \leq x \leq 1, \end{cases} \quad (18)$$

where  $\alpha \in (1, 2)$  is the order of the fractional derivative,  $q(x, t)$  is the source term, and diffusion coefficients  $d_\pm$  are nonnegative; i.e.,  $d_\pm \geq 0$ . In order to solve Eq. (18) numerically, let  $N$  and  $M$  be positive integers, and  $h = 1/(N+1)$  and  $\tau = 1/M$  be the sizes of spatial grid and time step, respectively. We define a spatial and temporal partition  $x_n = nh$  for  $n = 0, 1, \dots, N$  and  $t_m = m\tau$  for  $m = 0, 1, \dots, M$ . Let  $u_n^{(m)} = u(x_n, t_m)$  and  $q_n^{(m)} = q(x_n, t_m)$ . In [29], Meerschaert and Tadjeran proposed the shifted Grünwald approximations,

$$\begin{aligned} \frac{\partial^\alpha u(x_n, t_m)}{\partial_+ x^\alpha} &= \frac{1}{h^\alpha} \sum_{k=0}^{n+1} g_k^{(\alpha)} u_{n-k+1}^{(m)} + \mathcal{O}(h), \\ \frac{\partial^\alpha u(x_n, t_m)}{\partial_- x^\alpha} &= \frac{1}{h^\alpha} \sum_{k=0}^{N-n+2} g_k^{(\alpha)} u_{n+k-1}^{(m)} + \mathcal{O}(h), \end{aligned}$$

where the coefficients  $g_k^{(\alpha)}$  and the corresponding properties are given in [29, 30, Proposition 1]. By combining the implicit Euler scheme to discrete the Eq. (18), then the final numerical scheme

$$\frac{u_n^{(m+1)} - u_n^{(m)}}{\tau} = \frac{d_+}{h^\alpha} \sum_{k=0}^{n+1} g_k^{(\alpha)} u_{n-k+1}^{(m)} + \frac{d_-}{h^\alpha} \sum_{k=0}^{N-n+2} g_k^{(\alpha)} u_{n+k-1}^{(m)} + |u_n^{(m+1)}| + q_n^{(m+1)}. \quad (19)$$

By using the similar methods presented in [28], it is not difficult to prove that the numerical scheme (19) is unconditionally stable, which we will not pursue here. Let  $\mathbf{u}^{(m)} = (u_1^{(m)}, u_2^{(m)}, \dots, u_N^{(m)})^T$ ,  $\mathbf{q}^{(m)} = (q_1^{(m)}, q_2^{(m)}, \dots, q_N^{(m)})^T$  and  $I$  be the identity matrix with an appropriate size. Then the numerical scheme (19) can be written in the following matrix form

$$\left( I - \frac{\tau}{h^\alpha} (d_+ G_\alpha + d_- G_\alpha^T) \right) \mathbf{u}^{(m+1)} - |\mathbf{u}^{(m+1)}| = \mathbf{u}^{(m)} + \tau \mathbf{q}^{(m+1)},$$

where  $G_\alpha$  is a nonsymmetric Toeplitz matrix defined in [30]. According to Eq. (19), it implies that we need to handle a system of nonlinear equations like the AVE in (1) at each time step, i.e., there is a need for solving the AVEs with the form  $A\mathbf{u} - |\mathbf{u}| = \mathbf{f}$ , where  $\mathbf{f} = \mathbf{u}^{(m)} + \tau \mathbf{q}^{(m+1)}$  without the index  $m$  and  $A = I - \frac{\tau}{h^\alpha} (d_+ G_\alpha + d_- G_\alpha^T)$  is also a nonsymmetric Toeplitz matrix.

Next, we will employ the CSCS-based iteration methods to solve the above resulting AVEs, then the necessary condition for analyzing the convergence of the CSCS-based iteration method is that both circulant part  $C$  and skew-circulant part  $S$  of the coefficient matrix  $A$  are positive definite. In fact, we have already mentioned that both circulant part  $C$  and skew-circulant part  $S$  of the coefficient matrix  $A = I - \frac{\tau}{h\alpha}(d_+G_\alpha + d_-G_\alpha^T)$  are positive definite (see [31] for details) via the similarly analyzed methods in [32]. It means that exploiting the CSCS-based iteration methods for solving the resulting AVE is reasonable. Meanwhile, for simplicity, the vector  $\tilde{\mathbf{f}}$  is still set as  $\tilde{\mathbf{f}} = h^\alpha \mathbf{b}$ , where  $\mathbf{b}$  is the same as the previous contexts. At the same time, it is remarkable that HSS-based iteration methods are not suitable for Example 2 due to the Toeplitz coefficient matrix. Otherwise, it will lead to the complex computations for solving two sub-systems with the coefficient matrices  $\sigma I + H$  and  $\sigma I + S$ . On the other hand, Mangasarian modified the classical Newton iteration for handling the system of AVEs by introducing the auxiliary diagonal matrix  $\hat{D}(\mathbf{x}) = \partial|\mathbf{x}| = \text{diag}(\text{sign}(\mathbf{x}))$ , see [12] for details; then he established the generalized Newton iterative scheme with the initial guess  $\mathbf{x}^{(0)}$ ,

$$\mathbf{x}^{(k+1)} = \left( A - \hat{D}(\mathbf{x}^{(k)}) \right)^{-1} \tilde{\mathbf{f}}, \quad (20)$$

so it notes that we need to solve a system of linear equations with the coefficient matrix  $J^{(k)} = A - \hat{D}(\mathbf{x}^{(k)})$ , i.e. Eq. (20). Also, since the matrix  $J^{(k)}$  is a Toeplitz-plus-diagonal matrix, so there are no fast direct solvers for  $J^{(k)}\tilde{\mathbf{x}} = \tilde{\mathbf{f}}^1$ . Fortunately, it should mention that the matrix-vector product involving  $J^{(k)}$  can be implemented via FFTs due to having the Toeplitz part  $A$ . It tells us that the Krylov subspace methods, such as the restarted GMRES method [33] and the TFQMR method [34], can be compatibly exploited for solving  $J^{(k)}\tilde{\mathbf{x}} = \tilde{\mathbf{f}}$  at each iteration step. We conclude these iterative schemes as the GN-TFQMR method and the GN-GMRES( $m$ ) method, where  $m$  is the restarted number. In conclusion, we will compared the proposed CSCS-based iteration method with both the GN-GMRES( $m$ ) and GN-TFQMR methods for solving the resulting AVEs in Example 1. This idea about establishing two combined (inner-outer) iteration methods also follows recommendations in [35]. Numerical results are reported in the following tables under different values of  $\alpha, d_\pm$  and  $h = \tau$ .

Table 4: The optimal parameters  $\sigma_{opt}^*$  for Example 2.

$\alpha$	$(d_+, d_-)$	$\sigma_{opt}^*$	$\sigma_{opt}^*$					
			$n = 128$	$n = 256$	$n = 512$	$n = 1024$	$n = 2048$	$n = 4096$
1.2	(0.5, 0.8)	$\sigma_{\text{CSCS}}$	1.4499	1.5338	1.6233	1.7180	1.8175	1.9216
1.5	(0.6, 0.4)	$\sigma_{\text{CSCS}}$	2.7848	3.2426	3.7564	4.3094	4.8598	5.3131
1.8	(0.7, 0.3)	$\sigma_{\text{CSCS}}$	5.8492	6.9416	7.0941	15.3896	26.7749	46.6033

First of all, the optimal parameters  $\sigma_{CSCS}$  for Example 2 are listed in Table 4. It is remarked that with the increase of the matrix dimension  $n$ , the optimal parameters  $\sigma_{CSCS}$  are almost fixed or decreasing slightly for the cases of  $\alpha = 1.2$  and  $\alpha = 1.5$ . Since the case of  $\alpha = 1.8$  corresponding to the coefficient matrix  $A$  is very ill-conditioned, so the optimal parameters  $\sigma_{CSCS}$  are varied intensely. Moreover, in Tables 5-7, we report the numerical results with respect to the Picard-CSCS, nonlinear CSCS-like, GN-GMRES(5) and GN-TFQMR iterative methods. We also present the elapsed CPU time in seconds for the convergence (denoted as CPU) and the number of outer, inner and total iteration steps (outer and inner iterations only for Picard-CSCS, GN-GMRES(5) and GN-TFQMR) for the convergence (denoted as IT\_out, IT\_inn and IT, respectively).

Based on the numerical results in Tables 5-7, it is notable that these four iterative solvers, i.e., the Picard-CSCS and the nonlinear CSCS-like, can successfully obtain approximate solutions to the system of AVEs for all different matrix dimensions; whereas both the GN-GMRES(5) and the GN-TFQMR

<sup>1</sup>It is mainly because the displacement rank of the matrix  $J^{(k)}$  can take any value between 0 and  $n$ . Hence, fast Toeplitz solvers that are based on small displacement rank of matrices cannot be applied.

Table 5: Numerical results for Example 2 with different values of  $n$  ( $\gamma = 1.2$  and  $(c, d) = (0.5, 0.8)$ ).

Method		$n = 128$	$n = 256$	$n = 512$	$n = 1024$	$n = 2048$	$n = 4096$
Picard-CSCS	IT <sub>out</sub>	6	6	6	6	6	6
	IT <sub>inn</sub>	4.0000	4.0000	4.0000	4.1667	5.0000	5.0000
	IT	24	24	24	25	30	30
	CPU	0.0077	0.0098	0.0136	0.01998	0.0331	0.0585
CSCS-like	IT	12	13	14	15	16	18
	CPU	0.0032	0.0051	0.0078	0.0099	0.01776	0.0334
GN-GMRES(5)	IT <sub>out</sub>	max	max	max	max	max	max
	IT <sub>inn</sub>	—	—	—	—	—	—
	IT	Fail	Fail	Fail	Fail	Fail	Fail
	CPU	—	—	—	—	—	—
GN-TFQMR	IT <sub>out</sub>	max	max	max	max	max	max
	IT <sub>inn</sub>	—	—	—	—	—	—
	IT	Fail	Fail	Fail	Fail	Fail	Fail
	CPU	—	—	—	—	—	—

Table 6: Numerical results for Example 2 with different values of  $n$  ( $\gamma = 1.5$  and  $(c, d) = (0.6, 0.4)$ ).

Method		$n = 128$	$n = 256$	$n = 512$	$n = 1024$	$n = 2048$	$n = 4096$
Picard-CSCS	IT <sub>out</sub>	6	6	6	6	6	6
	IT <sub>inn</sub>	7.0000	8.1667	9.1667	10.6667	13.0000	14.3333
	IT	42	49	55	64	78	86
	CPU	0.0161	0.0189	0.0234	0.0335	0.0609	0.1307
CSCS-like	IT	24	29	35	43	54	69
	CPU	0.0063	0.0084	0.0119	0.0218	0.0486	0.1133
GN-GMRES(5)	IT <sub>out</sub>	max	max	max	max	max	max
	IT <sub>inn</sub>	—	—	—	—	—	—
	IT	Fail	Fail	Fail	Fail	Fail	Fail
	CPU	—	—	—	—	—	—
GN-TFQMR	IT <sub>out</sub>	max	max	max	max	max	max
	IT <sub>inn</sub>	—	—	—	—	—	—
	IT	Fail	Fail	Fail	Fail	Fail	Fail
	CPU	—	—	—	—	—	—

Table 7: Numerical results for Example 2 with different values of  $n$  ( $\gamma = 1.8$  and  $(c, d) = (0.7, 0.3)$ ).

Method		$n = 128$	$n = 256$	$n = 512$	$n = 1024$	$n = 2048$	$n = 4096$
Picard-CSCS	IT <sub>out</sub>	5	6	10	8	9	13
	IT <sub>inn</sub>	14.4000	15.0000	15.0000	15.0000	15.0000	15.0000
	IT	72	90	150	120	135	195
	CPU	0.0185	0.0252	0.0415	0.0505	0.0987	0.2817
CSCS-like	IT	59	86	146	117	117	118
	CPU	0.0101	0.0179	0.0386	0.0517	0.1007	0.1911
GN-GMRES(5)	IT <sub>out</sub>	max	max	max	max	max	max
	IT <sub>inn</sub>	—	—	—	—	—	—
	IT	Fail	Fail	Fail	Fail	Fail	Fail
	CPU	—	—	—	—	—	—
GN-TFQMR	IT <sub>out</sub>	max	max	max	max	max	max
	IT <sub>inn</sub>	—	—	—	—	—	—
	IT	Fail	Fail	Fail	Fail	Fail	Fail
	CPU	—	—	—	—	—	—

iterative methods fully fail to converge. It should be because the Newton-like iterative methods are usually sensitive to the initial guess and the accuracy of solving inner linear systems. When the matrix dimension  $n$  is increasing, the number of outer iteration steps are almost fixed or decreasing slightly for all iteration methods, whereas the number of inner iteration steps show the contrary phenomena for the cases with  $\alpha = 1.2$  and  $\alpha = 1.5$ . Meanwhile, the total CPU times and total iteration steps for both the Picard-CSCS and the nonlinear CSCS-like iterative methods are increasing quickly except the cases of  $\alpha = 1.8$  with  $n = 1024$  and  $n = 2048$ . On the other hand, from Tables 5-7, we also observe that both the nonlinear CSCS-like method is almost more competitive than the Picard-CSCS iteration methods in terms of the number of iteration steps and the CPU elapsed time for solving the system of AVEs. In particular, it is remarkable that the nonlinear CSCS-like method can use slightly less number of iteration steps to converge than the Picard-CSCS iterative solver, but the Picard-CSCS iterative solver can save a little elapsed CPU time with compared to the nonlinear CSCS-like iterative method in our implementations. However, it still concludes that the nonlinear CSCS-like iterative method is the first choice for solving the AVEs concerning in Example 2. At the same time, the Picard-CSCS iterative method can be viewed as a good alternative.

## 5 Conclusions

In this paper, we have proposed two CSCS-based methods for solving the system of AVEs with non-Hermitian Toeplitz structure. Two CSCS-based iterative methods are based on separable property of the linear term  $A\mathbf{x}$  and nonlinear term  $|\mathbf{x}| + \mathbf{b}$  as well as the circulant and skew-circulant splitting (CSCS) of involved non-Hermitian definite Toeplitz matrix  $A$ . By leveraging the smoothing approximate function, the locally convergence have been analysed. Further numerical experiments have shown that the Picard-CSCS and the nonlinear CSCS-like iteration methods are feasible and efficient nonlinear solvers for the AVE. Moreover, In particular, the nonlinear CSCS-like method often does better than the Picard-CSCS method to solve AVE is that the smoothing approximate function is introduced in the convergence analysis although is avoid in implement algorithm. Hence, to find a better theoretical proof for CSCS-like will be a topics and suitable accelerated techniques [16, 22, 36] in the future research.

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